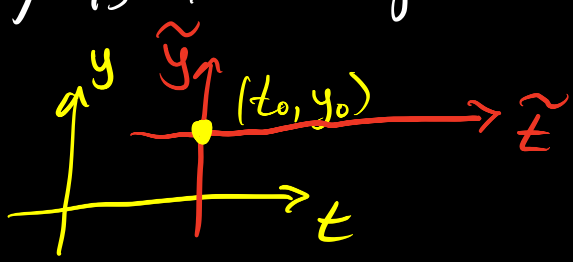
# § 2.8 The Existence and Uniqueness Theorem

**Thm** Existence and Uniqueness

If $f$ and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R: |t| \le a$, $|y| \le b$, then there is some interval $|t| \le h \le a$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \quad y(0) = 0 .$$

**Note** If we have the IVP $y' = f(t, y)$, $y(t_0) = y_0$, we can make a change of variables so that the initial point $(t_0, y_0)$ is the origin.



Now, we will examine elements of the existence/uniqueness theorem.

Suppose $y = \varphi(t)$ satisfies the IVP. Then $f(t,y) = f(t, \varphi(t))$, so $f$ only depends on $t$. Now,

$$y' = f(t,y)$$

$$y' = f(t, \varphi(t))$$

$$\int_0^t y' \, dt = \int_0^t f(s, \varphi(s)) \, ds$$

$$y(t) - y(0) = \int_0^t f(s, \varphi(s)) \, ds$$

$$y(t) = \int_0^t f(s, \varphi(s)) \, ds .$$

This expression for $y(t)$ involves the

integral of an unknown function $\phi$ and is called an integral equation.

★ The integral equation and IVP are equivalent ★

<u>Picard's iteration method (method of successive approximations)</u>

- Method for showing the integral equation has a unique solution

We generate a sequence of functions $\{\phi_n(t)\}$ as follows:

1. Choose $\phi_0$. Simplest choice is $\phi_0(t) = 0$.

2. We get $\phi_1$ by using $\phi_0$ in the integral equation

$$\varphi_1(t) = \int_0^t f(s, \varphi_0(s))\, ds$$

3. We get $\varphi_2$ by using $\varphi_1$

$$\varphi_2(t) = \int_0^t f(s, \varphi_1(s))\, ds$$

and so on. In general,

$$\varphi_{n+1}(t) = \int_0^t f(s, \varphi_n(s))\, ds .$$

We now have a sequence of functions. Each function in the sequence satisfies the the IC, but may not satisfy the DE. If at some point, say $n=k$, we have $\varphi_{k+1}(t) = \varphi_k(t)$, then $\varphi_k$ is a solution of the integral equation.

To prove the Existence/Uniqueness theorem we need to know:

1. Do all members of $\{\phi_n\}$ exist?

2. Does $\{\phi_n\}$ converge?

3. What are the properties of the limit function?

4. Is this the only solution?

<u>1. Do all members of $\{\phi_n\}$ exist?</u>

$f$ and $\frac{\partial f}{\partial y}$ are continuous in the rectangle $R: |t| \le a, |y| \le b$ so the danger is that $y = \phi_k(t)$ may be outside of R. We need to restrict $t$ to a smaller interval than $|t| \le a$.

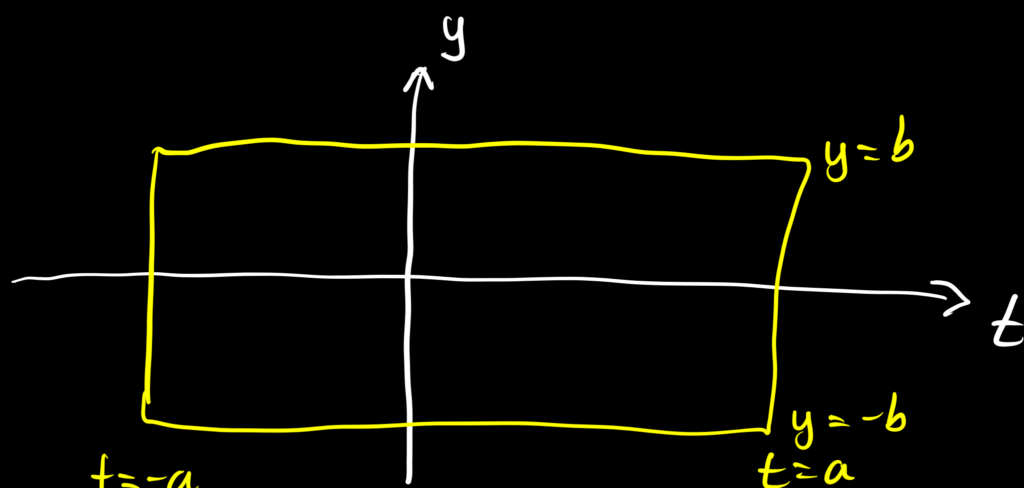
$f$ is continuous on a closed bounded region so there is constant $M > 0$ such that

$$|f(t,y)| \le M \text{ for } (t,y) \in R.$$

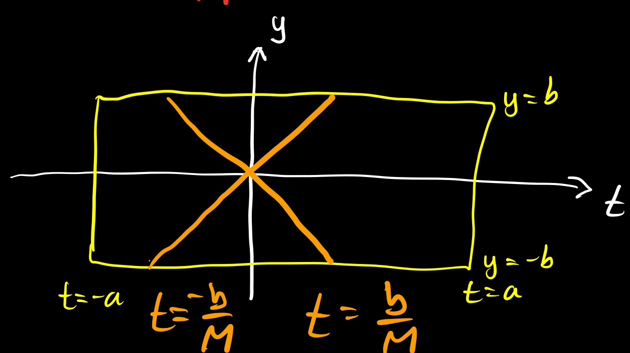
Now,

$$\phi_{k+1}' = f(t, \phi_k) \le M$$

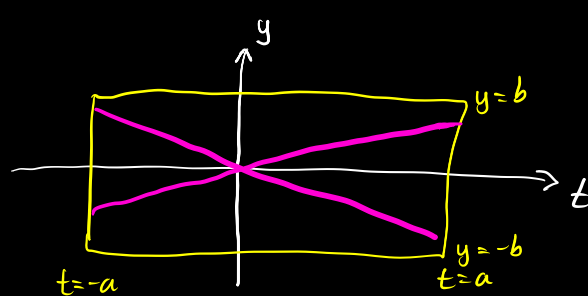
so the point $(t, \phi_{k+1}(t)) \in R$ as long as $|t| \le \frac{b}{M}$.




$\frac{b}{M} < a$




$\frac{b}{M} > a$




Choose $h = \min\{a, \frac{b}{M}\}$.

## 2. Does $\{\varphi_n\}$ converge?

We see that

$$\varphi_n(t) = \varphi_1 + (\varphi_2 - \varphi_1) + \cdots + (\varphi_n - \varphi_{n-1})$$

is the partial sum of

$$\varphi_1 + \sum_{k=1}^{\infty} (\varphi_{k+1} - \varphi_k).$$

Therefore, if the series converges, then the sequence $\{\varphi_n\}$ converges and we let

$$\varphi(t) = \lim_{n\to\infty} \varphi_n(t)$$

3. What are the properties of the limit function?

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We want to know if $\varphi$ is continuous.
If each $\varphi_n$ is continuous and $\{\varphi_n\}$ "converges uniformly" then $\varphi$ is continuous.

$$\varphi_{n+1}(t) = \int_0^t f(s, \varphi_n(s))\, ds$$

$$\varphi(t) = \lim_{n\to\infty} \int_0^t f(s, \varphi_n(s))\, ds$$

$$= \int_0^t \lim_{n\to\infty} f(s, \varphi_n(s))\, ds$$ Since $\varphi_n$ converges uniformly

$$= \int_0^t f(s, \lim_{n\to\infty} \varphi_n(s))\, ds$$ Since $f$ is continuous

$$= \int_0^t f(s, \varphi(s))\, ds$$

4. Is this the only solution?

Suppose $y_1 = \varphi(t)$ and $y_2 = \psi(t)$ are both solutions. Then

$$|y_1 - y_2| = |\varphi(t) - \psi(t)|$$

$$= \left| \int_0^t f(s,\varphi(s))\,ds - \int_0^t f(s,\psi(s))\,ds \right|$$

$$= \left| \int_0^t f(s,\varphi(s)) - f(s,\psi(s))\,ds \right|$$

$$\leq \int_0^t \left| f(s,\varphi(s)) - f(s,\psi(s)) \right| ds$$

$$\leq \int_0^t L \left| \varphi(s) - \psi(s) \right| ds$$

for some constant $L$. (Lipschitz constant)

Let

$$A(t) = \int_0^t |\varphi(s) - \psi(s)|\,ds.$$

Then

$$A(0) = 0$$

$$A(t) \geq 0 \quad \text{for} \quad t \geq 0$$

and

$$A'(t) = |\varphi(t) - \psi(t)|.$$

Now,

$$|\varphi(t) - \psi(t)| \le L \int_0^t |\varphi(s) - \psi(s)|\, ds$$

$$A'(t) \le L A(t)$$

$$A'(t) - L A(t) \le 0 .$$

Multiply by $e^{-Lt}$, we get

$$\frac{d}{dt}\left( e^{-Lt} A(t) \right) \le 0$$

so $$e^{-Lt} A(t) \le 0 .$$

Hence $$A(t) \le 0 .$$

We now have that

$A(0) = 0$ and $A(t) \ge 0$ for $t \ge 0$.

AND

$A(t) \leq 0$ for $t \geq 0$.

So $A(t) = 0$ for $t \geq 0$.

Hence, $A'(t) = 0$

so

$|\varphi(t) - \psi(t)| = 0$

i.e. $\varphi(t) = \psi(t)$ for $t \geq 0$.

## §2.9 First-Order Difference Equations

So far, we have considered continuous models but now we explore discrete models which lead to difference equations.

A first-order difference equation is of the form
$$y_{n+1} = f(n, y_n), \quad n = 0, 1, 2, \ldots .$$

It is first-order since $y_{n+1}$ only depends on $y_n$.

The difference eqn is linear if $f$ is a linear function of $y_n$, otherwise it is nonlinear.

A solution is a sequence of numbers $y_0, y_1, \ldots$ that satisfy the equation for each $n$.

The initial condition $y_0 = \alpha$, specifies the first term of the solution sequence.

## Equilibrium Solutions

Suppose $\quad y_{n+1} = f(y_n), \quad n = 0, 1, 2, \ldots$

where $f$ depends only on $y_n$. Then

$$y_1 = f(y_0)$$

$$y_2 = f(y_1) = f(f(y_0))$$

$$y_3 = f(y_2) = f(f(f(y_0))) = f^3(y_0)$$

and so

$$y_n = f^n(y_0)$$

We may want to know what happens to $y_n$ as $n \to \infty$. If $y_n$ has the same value for all $n$, $y_n$ is called an equilibrium solution. Such a solution satisfies

$$y_n = f(y_n) .$$

<u>Linear Equations</u>

Consider the difference equation

$$y_{n+1} = \rho y_n + b_n , \quad n = 0, 1, 2, \ldots$$

First, let's solve the equation in terms of the initial value $y_0$. We have

$$y_1 = \rho y_0 + b_0$$

$$y_2 = \rho y_1 + b_1 = \rho(\rho y_0 + b_0) + b_1$$
$$= \rho^2 y_0 + \rho b_0 + b_1$$

$$y_3 = \rho y_2 + b_2$$
$$= \rho(\rho^2 y_0 + \rho b_0 + b_1) + b_2$$
$$= \rho^3 y_0 + \rho^2 b_0 + \rho b_1 + b_2$$

and so

$$y_n = \rho^n y_0 + \rho^{n-1} b_0 + \cdots + \rho b_{n-2} + b_{n-1}$$
$$= \rho^n y_0 + \sum_{j=0}^{n-1} \rho^{n-1-j} b_j .$$

Suppose $b_n = b \neq 0$ for all $n$, then

$$y_n = \rho^n y_0 + b \sum_{j=0}^{n-1} \rho^{n-1-j}$$

$$= \rho^n y_0 + b \frac{1-\rho^n}{1-\rho}$$

$$= \rho^n \left( y_0 - \frac{b}{1-\rho} \right) + \frac{b}{1-\rho}$$

Now, let's determine the long-time behavior of $y_n$.

1. $|\rho| < 1 \Rightarrow y_n \to \frac{b}{1-\rho}$

2. $|\rho| > 1$

   a. $y_0 = \frac{b}{1-\rho} \Rightarrow y_n = \frac{b}{1-\rho}$ for all $n$

   b. $y_0 \neq \frac{b}{1-\rho} \Rightarrow$ no limit

3. $\rho = -1 \Rightarrow$ no limit

4. $\rho = 1$, we need to go back to the difference equation

$$y_{n+1} = \rho y_n + b$$
$$= y_n + b$$

$$\Rightarrow y_n \to \infty \quad \text{as} \quad n \to \infty$$

## Nonlinear Equations

Logistic difference equation

$$y_{n+1} = \rho y_n \left(1 - \frac{y_n}{K}\right)$$

or

$$u_{n+1} = \rho u_n (1 - u_n)$$

where $u_n = \frac{y_n}{K}$.

Let's find equilibrium solutions

$$u_n = \rho u_n (1 - u_n)$$

$$\rho u_n^2 + u_n(1-\rho) = 0$$

$$u_n(\rho u_n + 1 - \rho) = 0$$

$$u_n = 0 \qquad u_n = \frac{\rho - 1}{\rho}$$

## Cobweb / Stairstep diagram